A formula for the First Eigenvalue of the Dirac Operator on Compact Spin Symmetric Spaces

Jean-Louis Milhorat

A FORMULA FOR THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON COMPACT SPIN SYMMETRIC SPACES

JEAN-LOUIS MILHORAT

Abstract. Let G/K be a simply connected spin compact inner irreducible symmetric space, endowed with the metric induced by the Killing form of G sign-changed. We give a formula for the square of the first eigenvalue of the Dirac operator in terms of a root system of G. As an example of application, we give the list of the first eigenvalues for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure.

1. Introduction

Let G/K be a compact, simply-connected, n-dimensional irreducible symmetric space with G compact and simply-connected, endowed with the metric induced by the Killing form of G sign-changed. Assume that G and K have same rank and that G/K has a spin structure. In a previous paper, cf. [Mil04], we proved that the first eigenvalue λ of the Dirac operator verifies

$$\lambda^2 = 2 \min_{1\leq k\leq p} \|\beta_k\|^2 + n/8\,, \tag{1}$$

where β_k, $k = 1, \ldots, p$, are the K-dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of K, and where $\|\cdot\|$ is the norm associated to the scalar product induced by the Killing form of G.

The proof was based on a lemma of R. Parthasarathy in [Par71], which allows to express the result in the following way.

Let T be a fixed common maximal torus of G and K. Let Φ be the set of non-zero roots of G with respect to T. Let Φ_G^+ be the set of positive roots of G, Φ_K^+ be the set of positive roots of K, with respect to a fixed lexicographic ordering in Φ. Let δ_G, (resp. δ_K) be the half-sum of the positive roots of G, (resp. K). Then the square of the first eigenvalue of the Dirac operator is given by

$$\lambda^2 = 2 \min_{w\in W} \|w\cdot\delta_G - \delta_K\|^2 + n/8\,, \tag{2}$$

where W is the subset of the Weyl group W_G defined by

$$W := \{w \in W_G\ ;\ w\cdot\Phi_G^+ \supset \Phi_K^+\}\,. \tag{3}$$

In order to avoid the determination of the subset W for applications, we prove in the following that the square of the first eigenvalue of the Dirac operator is indeed given by

$$\boxed{\lambda^2 = 2 \min_{w\in W_G} \|w\cdot\delta_G - \delta_K\|^2 + n/8\,.} \tag{4}$$

We then give a different expression to use the formula for explicit computations. We obtain

$$\boxed{\lambda^2 = 2\,\|\delta_G - \delta_K\|^2 + 4\sum_{\theta\in\Lambda} < \theta, \delta_K > + n/8\,,} \tag{5}$$

where Λ is the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, < \theta, \delta_K > \; < 0\}\,.$$

As an example of application of the above formula, we obtain the list of the first eigenvalues of the Dirac operator for the spin compact irreducible symmetric spaces endowed with a quaternion-Kähler structure. By definition, a Riemannian manifold has a quaternion-Kähler structure if its holonomy group is contained in the group $\mathrm{Sp}_m\mathrm{Sp}_1$. In [Wol65], J. Wolf gave the following classification of compact quaternion-Kähler symmetric spaces:

G	K	G/K	dim G/K	Spin structure (cf. [CG88])
Sp_{m+1}	$\mathrm{Sp}_m \times \mathrm{Sp}_1$	Quaternionic projective space $\mathbb{H}P^m$	$4m\,(m \geq 1)$	Yes (unique)
SU_{m+2}	$S(\mathrm{U}_m \times \mathrm{U}_2)$	Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2})$	$4m\,(m \geq 1)$	iff m even unique in that case
Spin_{m+4}	$\mathrm{Spin}_m\mathrm{Spin}_4$	Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4})$	$4m\,(m \geq 3)$	iff m even, unique in that case
G_2	SO_4		8	Yes (unique)
F_4	$\mathrm{Sp}_3\mathrm{SU}_2$		28	No
E_6	$\mathrm{SU}_6\mathrm{SU}_2$		40	Yes (unique)
E_7	$\mathrm{Spin}_{12}\mathrm{SU}_2$		64	Yes (unique)
E_8	$\mathrm{E}_7\mathrm{SU}_2$		112	Yes (unique)

Note furthermore that all the symmetric spaces in that list are “inner”.
Endowing each symmetric space with the metric induced by the Killing form of G sign-changed, we obtain the following table

G/K	Square of the first eigenvalue of D
$\mathbb{H}P^n = \mathrm{Sp}_{m+1}/(\mathrm{Sp}_m \times \mathrm{Sp}_1)$	$\frac{m+3}{m+2}\frac{m}{2} = \frac{m+3}{m+2}\frac{\mathrm{Scal}}{4}$
$\mathrm{Gr}_2(\mathbb{C}^{m+2}) = \mathrm{SU}_{m+2}/S(\mathrm{U}_m \times \mathrm{U}_2)$ (m even)	$\frac{m+4}{m+2}\frac{m}{2} = \frac{m+4}{m+2}\frac{\mathrm{Scal}}{4}$
$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4}) = \mathrm{Spin}_{m+4}/\mathrm{Spin}_m\mathrm{Spin}_4$ (m even)	$\frac{m^2+6m-4}{m(m+2)}\frac{m}{2} = \frac{m^2+6m-4}{m(m+2)}\frac{\mathrm{Scal}}{4}$
$\mathrm{G}_2/\mathrm{SO}_4$	$\frac{3}{2} = \frac{3}{2}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_6/(\mathrm{SU}_6\mathrm{SU}_2)$	$\frac{41}{6} = \frac{41}{30}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_7/(\mathrm{Spin}_{12}\mathrm{SU}_2)$	$\frac{95}{9} = \frac{95}{72}\frac{\mathrm{Scal}}{4}$
$\mathrm{E}_8/(\mathrm{E}_7\mathrm{SU}_2)$	$\frac{269}{15} = \frac{269}{210}\frac{\mathrm{Scal}}{4}$

TABLE I

The result was already known for quaternionic projective spaces $\mathbb{H}P^n$, [Mil92], for the Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{m+2})$, [Mil98], and for the symmetric space $\mathrm{G}_2/\mathrm{SO}_4$, [See99]. Up to our knowledge, the other results are new.

2. Proof of formula (4)

With the notations of the introduction, and since the scalar product is W_G-invariant, one has for any $w \in W_G$

$$\|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 < w \cdot \delta_G, \delta_K > , \tag{6}$$

hence

$$\min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W} < w \cdot \delta_G, \delta_K >, ,$$

and

$$\min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \max_{w \in W_G} < w \cdot \delta_G, \delta_K >, .$$

So we have to prove that

$$\max_{w\in W} < w\cdot\delta_G, \delta_K >= \max_{w\in W_G} < w\cdot\delta_G, \delta_K > . \tag{7}$$

Let

$$\Pi_G := \{\theta_1, \ldots, \theta_r\} \subset \Phi_G^+ , \tag{8}$$

be the set of G-simple roots and let

$$\Pi_K := \{\theta_1', \ldots, \theta_l'\} \subset \Phi_K^+ , \tag{9}$$

be the set of K-simple roots.
Let $w_0 \in W_G$ such that

$$< w_0\cdot\delta_G, \delta_K >= \max_{w\in W_G} < w\cdot\delta_G, \delta_K > . \tag{10}$$

Suppose that $w_0 \notin W$. Then we claim that there exists a K-simple root θ_i' such that $w_0^{-1}\cdot\theta_i' \notin \Phi_G^+$. Otherwise, if for any K-simple root θ_i', $w_0^{-1}\cdot\theta_i' \in \Phi_G^+$, then since any K-positive root is a linear combination with non-negative coefficients of K-simple roots, we would have $\forall\theta' \in \Phi_K^+$, $w_0^{-1}\cdot\theta' \in \Phi_G^+$, contradicting the assumption made on w_0.
Now let σ_i' be the reflection across the hyperplane ${\theta_i'}^{\perp}$. Since $\sigma_i'\cdot\delta_K = \delta_K - \theta_i'$, (cf. for instance Corollary of Lemma B, §10 .3 in [Hum72]), one gets by the W_G-invariance of the scalar product

$$\begin{aligned} < \sigma_i' w_0\cdot\delta_G, \delta_K > &=< w_0\cdot\delta_G, \sigma_i'\cdot\delta_K >=< w_0\cdot\delta_G, \delta_K - \theta_i' > \\ &=< w_0\cdot\delta_G, \delta_K > - < \delta_G, w_0^{-1}\cdot\theta_i' > . \end{aligned}$$

But since $w_0^{-1}\cdot\theta_i'$ is a negative root of G, one has

$$w_0^{-1}\cdot\theta_i' = -\sum k_j\,\theta_j\,, \quad k_j \in \mathbb{N}\,.$$

Since for any G-simple root θ_j, $\sigma_j\cdot\delta_G = \delta_G - \theta_j$, where σ_j is the reflection across the hyperplane ${\theta_j}^{\perp}$, one has $< \theta_j, \delta_G >= 2 < \theta_j, \theta_j > > 0$, so

$$- < \delta_G, w_0^{-1}\cdot\theta_i' >= \sum k_j < \delta_G, \theta_j > > 0\,,$$

hence

$$< \sigma_i' w_0\cdot\delta_G, \delta_K > > < w_0\cdot\delta_G, \delta_K > ,$$

but that is in contradiction with the definition (10) of w_0, hence $w_0 \in W$ and

$$\max_{w\in W_G} < w\cdot\delta_G, \delta_K >=< w_0\cdot\delta_G, \delta_K >\le \max_{w\in W} < w\cdot\delta_G, \delta_K >\le \max_{w\in W_G} < w\cdot\delta_G, \delta_K > ,$$

hence the result.

3. Proof of formula (5)

In order to obtain the formula we will use the following result

Lemma 3.1. *For any element w of the Weyl group W_G*

$$w\cdot\delta_G = \delta_G - \sum_{\theta\in\Phi_G^+} k_\theta\,\theta\,, \quad k_\theta = 0 \text{ or } 1\,.$$

Proof. Let $w \in W_G$. With the same notations as in the above proof, we write w in reduced form

$$w = \sigma_{i_1} \cdots \sigma_{i_k}\,, \tag{11}$$

where σ_i is the reflection across the hyperplane $\theta_i^{\perp}$, $\theta_i \in \Pi_G$, and k is minimal.
Since $\sigma_{i_k} \cdot \delta_G = \delta_G - \theta_{i_k}$, one has

$$w \cdot \delta_G = \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\sigma_{i_k} \cdot \delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-1}} (\theta_{i_k})\,.$$

Now, since the expression of w is reduced, $w(\theta_{i_k})$ is a negative root, cf. for instance corollary of Lemma C, § 10.3 in [Hum72]. But $w(\theta_{i_k}) = -\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$, hence $\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\theta_{i_k})$ is a positive root.
Now the element $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \in W_G$ is written in reduced form, otherwise the expression (11) of w would not be reduced. Hence we may conclude as above that

$$\sigma_{i_1} \cdots \sigma_{i_{k-1}} (\delta_G) = \sigma_{i_1} \cdots \sigma_{i_{k-2}} (\delta_G) - \sigma_{i_1} \cdots \sigma_{i_{k-2}} (\theta_{i_{k-1}})\,,$$

where $\sigma_{i_1} \cdots \sigma_{i_{k-2}}(\theta_{i_{k-1}})$ is a positive root.
Proceeding inductively we get

$$w \cdot \delta_G = \delta_G - \sum_{\theta \in \Phi_G^+} k_\theta\, \theta\,, \quad k_\theta \in \mathbb{N}\,.$$

In order to conclude, we have to prove that if a G-positive root θ appears in the above sum, then it appears only once.
Suppose that a G-positive root appears at least twice in the above sum, then there exist two integers p and q, $1 \le p < q \le k-1$ such that

$$\sigma_{i_1} \cdots \sigma_{i_p} (\theta_{i_{p+1}}) = \sigma_{i_1} \cdots \sigma_{i_q} (\theta_{i_{q+1}})\,.$$

applying $\sigma_{i_{p+1}} \sigma_{i_p} \cdots \sigma_{i_1}$ to the two members of the above equation, we get

$$\begin{cases} -\theta_{i_{p+1}} = \sigma_{i_{p+2}} \cdots \sigma_{i_q} (\theta_{i_{q+1}})\,, & \text{if } p+1 < q, \\ -\theta_{i_q} = \theta_{i_{q+1}}\,, & \text{if } p+1 = q. \end{cases}$$

So we get a contradiction, even in the first case, since $\sigma_{i_{p+2}} \cdots \sigma_{i_q} \sigma_{i_{q+1}} \in W_G$ is expressed in reduced form (otherwise the expression (11) of w would not be reduced), hence $\sigma_{i_{p+2}} \cdots \sigma_{i_q}(\theta_{i_{q+1}})$ is a positive root. □

From the above result we deduce

Lemma 3.2. *Let Λ be the set*

$$\Lambda := \{\theta \in \Phi_G^+\,;\, <\theta, \delta_K> \; < 0\}\,. \tag{12}$$

One has

$$\max_{w \in W_G} < w \cdot \delta_G, \delta_K > = < \delta_G, \delta_K > - \sum_{\theta \in \Lambda} < \theta, \delta_K >\,,$$

(setting $\sum_{\theta \in \Lambda} < \theta, \delta_K > = 0$, if $\Lambda = \emptyset$).

Proof. Suppose $\Lambda \neq \emptyset$. We first prove that there exists $w_0 \in W_G$ such that

$$w_0 \cdot \delta_G = \delta_G - \sum_{\theta \in \Lambda} \theta\,.$$

Let

$$\Phi_n^+ := \Phi_G^+ \backslash \Phi_K^+\,.$$

We first remark that any root in Λ belongs to Φ_n^+. Otherwise, if there exists $\theta \in \Lambda \cap \Phi_K^+$, then since θ is a combination with non-negative coefficients of simple

K-roots, and since $< \delta_K, \theta'_i >> 0$, for any K-simple root θ'_i, we would have $< \delta_K, \theta >\geq 0$, contradicting the fact that $\theta \in \Lambda$.
Now, consider

$$\delta_n := \frac{1}{2} \sum_{\theta \in \Phi_n^+} \theta = \delta_G - \delta_K \,.$$

Then

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \left(\delta_n - \sum_{\theta \in \Lambda} \theta \right) .$$

But,

$$\beta := \delta_n - \sum_{\theta \in \Lambda} \theta \,,$$

is a weight of the decomposition of the spin representation under the action of K, cf. § 2 in [Par71]: the weights are just the elements of the form $\delta_n - \sum_{\theta \in \Upsilon} \theta$, where Υ is a subset of Φ_n^+.
In fact β is the highest weight of an irreducible component in the decomposition, otherwise we would have

$$\beta + \alpha = \delta_n - \sum_{\theta \in \Upsilon} \theta \,,$$

where α is a K-positive root and Υ is a subset of Φ_n^+.
Hence setting $\Lambda' := \Lambda \backslash \Upsilon$ and $\Upsilon' := \Upsilon \backslash \Lambda$, we would have

$$- \sum_{\theta \in \Lambda'} \theta + \alpha = - \sum_{\theta \in \Upsilon'} \theta \,.$$

But since $\Lambda' \subset \Lambda$ and α is a K-positive root

$$< - \sum_{\theta \in \Lambda'} \theta + \alpha, \delta_K >> 0 \,,$$

whereas since $\Upsilon' \subset \Phi_n^+ \backslash \Lambda$

$$< - \sum_{\theta \in \Upsilon'} \theta, \delta_K > \leq 0 \,,$$

hence a contradiction.
Now by the result of lemma 2.2 in [Par71], any highest weight in the decomposition of the spin representation has the form

$$w \cdot \delta_G - \delta_K \,,$$

where w belongs to the subset W of W_G defined in (3). Hence there exists a $w_0 \in W$ such that

$$\beta = w_0 \cdot \delta_G - \delta_K \,,$$

hence

$$\delta_G - \sum_{\theta \in \Lambda} \theta = \delta_K + \beta = w_0 \cdot \delta_G \,,$$

hence the result.

Now let w be any element in W_G. By the above lemma,

$$\begin{aligned} w\cdot\delta_G &= \delta_G - \sum_{\theta\in\Phi_G^+} k_\theta\,\theta\,, \qquad\qquad k_\theta = 0 \text{ or } 1\,,\\ &= \delta_G - \sum_{\theta\in\Lambda} k_\theta\,\theta - \sum_{\theta\in\Phi_G^+\setminus\Lambda} k_\theta\,\theta\,. \end{aligned}$$

Hence by the definition of Λ

$$< w\cdot\delta_G,\delta_K > \leq < \delta_G - \sum_{\theta\in\Lambda} k_\theta\,\theta, \delta_K > \leq < \delta_G - \sum_{\theta\in\Lambda}\theta,\delta_K > \ .$$

Thus

$$\begin{aligned} \max_{w\in W_G} < w\cdot\delta_G,\delta_K > &\leq < \delta_G,\delta_K > - \sum_{\theta\in\Lambda} < \theta,\delta_K > = < w_0\cdot\delta_G,\delta_K >\\ &\leq \max_{w\in W_G} < w\cdot\delta_G,\delta_K >\,, \end{aligned}$$

hence the result. □

Now going back to formula (4), we get immediately from (6)

Corollary 3.3. *The first eigenvalue λ of the Dirac operator verifies*

$$\lambda^2 = 2\,\|\delta_G-\delta_K\|^2 + 4\sum_{\theta\in\Lambda} < \theta,\delta_K > + n/8\,.$$

4. Proof of the results of Table I

In the following, we note for any integer $n\geq 1$, $(e_1,\dots,e_n)$, the standard basis of $\mathbb{K}^n$, $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The space of (n,n) matrices with coefficients in $\mathbb{K}$ is denoted by $\mathrm{M}_n(\mathbb{K})$.

4.1. **Quaternionic projective spaces $\mathbb{H}P^n$.** Here $G=\mathrm{Sp}_{m+1}$ and $K=\mathrm{Sp}_m\times\mathrm{Sp}_1$. The decomposition of the spin representation into irreducible components under the action of K is given in [Mil92], so we may conclude with formula (1). However the result may be also simply concluded with formula (5).
The space $\mathbb{H}^{n+1}$ is viewed as a right vector space on $\mathbb{H}$ in such a way that G may be identified with the group

$$\left\{A\in\mathrm{M}_{m+1}(\mathbb{H})\,;\,{}^t\!AA=I_{m+1}\right\}\ ,$$

acting on the left on $\mathbb{H}^{n+1}$ in the usual way. The group K is identified with the subgroup of G defined by

$$\left\{A\in\mathrm{M}_{m+1}(\mathbb{H})\,;\,A=\begin{pmatrix}B&0\\0&q\end{pmatrix}\ ,\ {}^t\!BB=I_m\ ,\ q\in\mathrm{Sp}_1\right\}\ .$$

Let T be the common torus of G and K

$$T:=\left\{\begin{pmatrix}\mathrm{e}^{\mathbf{i}\beta_1}&&\\&\ddots&\\&&\mathrm{e}^{\mathbf{i}\beta_{m+1}}\end{pmatrix}\ ,\ \beta_1,\dots,\beta_{m+1}\in\mathbb{R}\right\}\ ,$$

where

$$\forall\beta\in\mathbb{R}\,,\quad \mathrm{e}^{\mathbf{i}\beta}:=\cos(\beta)+\sin(\beta)\,\mathbf{i}\,,$$

$(1,\mathbf{i},\mathbf{j},\mathbf{k})$ being the standard basis of $\mathbb{H}$.

The Lie algebra of T is

$$\mathfrak{T} = \left\{ \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} \; ; \; \beta_1, \beta_2, , \dots, \beta_{m+1} \, \in \mathbb{R} \right\} .$$

We denote by $(x_1, \dots, x_{m+1})$ the basis of $\mathfrak{T}^*$ given by

$$x_k \cdot \begin{pmatrix} \mathbf{i}\beta_1 & & \\ & \ddots & \\ & & \mathbf{i}\beta_{m+1} \end{pmatrix} = \beta_k \, .$$

A vector $\mu \in i\,\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+1} \mu_k \, \widehat{x}_k$, in the basis $(\widehat{x}_k \equiv i\, x_k)_{k=1,\dots,m+1}$, is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1}) \, .$$

The restriction to $\mathfrak{T}$ of the Killing form B of G is given by

$$\forall X \in \mathfrak{T} \, , \forall Y \in \mathfrak{T} \, , \quad B(X,Y) = 4\,(m+2)\, \Re\big(\operatorname{tr}(X\,Y)\big) \, .$$

It is easy to verify that the scalar product on $i\,\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$\forall \mu = (\mu_1, \dots, \mu_{m+1}) \in i\,\mathfrak{T}^* \, , \forall \mu' = (\mu'_1, \dots, \mu'_{m+1}) \, \in i\,\mathfrak{T}^* \, ,$$

(13)
$$< \mu, \mu' > = \frac{1}{4(m+2)} \sum_{k=1}^{m+1} \mu_k \, \mu'_k \, .$$

Now, considering the decomposition of the complexified Lie algebra of G under the action of T, it is easy to verify that T is a common maximal torus of G and K, and that the respective roots are given by

$$\begin{cases} \pm(\widehat{x}_i + \widehat{x}_j) \, , \\ \pm(\widehat{x}_i - \widehat{x}_i) \, , \end{cases} \quad 1 \le i < j \le m+1 \, , \qquad \pm\, 2\, \widehat{x}_i \, , \; 1 \le i \le m+1 \qquad \text{for } G \, ,$$

$$\begin{cases} \pm(\widehat{x}_i + \widehat{x}_j) \, , \\ \pm(\widehat{x}_i - \widehat{x}_j) \, , \end{cases} \quad 1 \le i < j \le m \, , \qquad \pm\, 2\, \widehat{x}_i \, , \; 1 \le i \le m+1 \qquad \text{for } K \, .$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \begin{cases} \widehat{x}_i + \widehat{x}_j \, , \\ \widehat{x}_i - \widehat{x}_j \, , \end{cases} \quad 1 \le i \le j \le m+1 \, ; \; 2\, \widehat{x}_i \, , \; 1 \le i \le m+1 \right\} ,$$

and

$$\Phi_K^+ = \left\{ \begin{cases} \widehat{x}_i + \widehat{x}_j \, , \\ \widehat{x}_i - \widehat{x}_j \, , \end{cases} \quad 1 \le i \le j \le m \, ; \; 2\, \widehat{x}_i \, , \; 1 \le i \le m+1 \right\} .$$

Then

$$\delta_G = \sum_{k=1}^{m+1} (m+2-k)\, \widehat{x}_k = (m+1, m, \dots, 2, 1) \, ,$$

and

$$\delta_K = \sum_{k=1}^{m}(m+1-k)\,\widehat{x}_k + \widehat{x}_{m+1} = (m, m-1, \dots, 1, 1)\,.$$

Hence

$$\delta_G - \delta_K = \sum_{k=1}^{m} \widehat{x}_k = (1, 1, \dots, 1, 0)\,,$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4(m+2)}\,.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+ \,;\, <\theta, \delta_K> \;< 0\}\,,$$

is empty, hence by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2(m+2)} + \frac{m}{2} = \frac{m+3}{m+2}\,\frac{m}{2}\,.$$

4.2. **Grassmannians $\mathbf{Gr_2}(\mathbb{C}^{\boldsymbol{m+2}})$, $\boldsymbol{m}$ even $\boldsymbol{\geq 2}$.** Here $G = \mathrm{SU}_{m+2}$ and K is the subgroup $S(\mathrm{U}_m \times \mathrm{U}_2)$ defined below. Here again, the decomposition into irreducible components of the spin representation under the action of K is known, [Mil98], hence the result may be obtained from formula (1). However the result may be also simply concluded with formula (5).
The group G is identified with

$$\left\{A \in \mathrm{M}_{m+2}(\mathbb{C})\,;\, {}^t\!AA = I_{m+2} \text{ and } \det A = 1\right\}\,.$$

The group K is the group

$$S(\mathrm{U}_m \times \mathrm{U}_2) = \left\{A \in \mathrm{M}_{m+2}(\mathbb{C})\,;\, A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}\,,\, B \in \mathrm{U}_m\,,\, C \in \mathrm{U}_2\,;\, \det A = 1\right\}\,.$$

Let T be the common torus of G and K

$$T := \left\{ \begin{pmatrix} \mathrm{e}^{i\beta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\beta_{m+2}} \end{pmatrix}\,,\, \beta_1, \dots, \beta_{m+2} \in \mathbb{R}\,,\, \sum_{k=1}^{m+2} \beta_k = 0 \right\}\,.$$

The Lie algebra of T is

$$\mathfrak{T} = \left\{ \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix}\,;\, \beta_1, \beta_2, \dots, \beta_{m+2} \in \mathbb{R}\,,\, \sum_{k=1}^{m+2} \beta_k = 0 \right\}\,.$$

We denote by $(x_1, \dots, x_{m+1})$ the basis of $\mathfrak{T}^*$ given by

$$x_k \cdot \begin{pmatrix} i\beta_1 & & \\ & \ddots & \\ & & i\beta_{m+2} \end{pmatrix} = \beta_k\,.$$

A vector $\mu \in i\,\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{m+1} \mu_k\,\widehat{x}_k$, in the basis $(\widehat{x}_k \equiv i\,x_k)_{k=1,\dots,m+1}$, is denoted by

$$\mu = (\mu_1, \mu_2, \dots, \mu_{m+1})\,.$$

The restriction to $\mathfrak{T}$ of the Killing form B of G is given by

$$\forall X \in \mathfrak{T}\,,\,\forall Y \in \mathfrak{T}\,,\quad B(X,Y) = 2\,(m+2)\,\Re\big(\operatorname{tr}(X\,Y)\big)\,.$$

It is easy to verify that the scalar product on $i\,\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$\text{(14)}\quad \forall \mu = (\mu_1,\dots,\mu_{m+1}) \in i\,\mathfrak{T}^*\,,\,\forall \mu' = (\mu'_1,\dots,\mu'_{m+1}) \in i\,\mathfrak{T}^*\,,$$
$$< \mu,\mu' >= \frac{1}{2(m+2)}\sum_{k=1}^{m+1}\mu_k\,\mu'_k - \frac{1}{2(m+2)^2}\left(\sum_{k=1}^{m+1}\mu_k\right)\left(\sum_{k=1}^{m+1}\mu'_k\right)\,.$$

Considering the decomposition of the complexified Lie algebra of G under the action of T, it is easy to verify that T is a common maximal torus of G and K, and that the respective roots are given by

$$\pm(\widehat{x}_i - \widehat{x}_j)\,,\ 1 \le i < j \le m+1\,,\qquad \pm\left(\widehat{x}_i + \sum_{k=1}^{m+1}\widehat{x}_k\right),1 \le i \le m+1\,,\quad \text{for } G\,,$$

$$\pm(\widehat{x}_i - \widehat{x}_j)\,,\ 1 \le i < j \le m\,,\qquad \pm\left(\widehat{x}_{m+1} + \sum_{k=1}^{m+1}\widehat{x}_k\right),\qquad \text{for } K\,.$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{\widehat{x}_i - \widehat{x}_j\ ,\ 1 \le i \le m+1\,;\ \widehat{x}_i + \sum_{k=1}^{m+1}\widehat{x}_k\ ,\ 1 \le i \le m+1\right\}\,,$$

and

$$\Phi_K^+ = \left\{\widehat{x}_i - \widehat{x}_j\ ,\ 1 \le i \le m\,;\ \widehat{x}_{m+1} + \sum_{k=1}^{m+1}\widehat{x}_k\right\}\,.$$

Then

$$\delta_G = \sum_{k=1}^{m+1}(m+2-k)\,\widehat{x}_k = (m+1,m,\dots,2,1)\,,$$

and

$$\delta_K = \frac{1}{2}\left(\sum_{k=1}^{m}(m+2-2k)\,\widehat{x}_k + 2\,\widehat{x}_{m+1}\right) = \frac{1}{2}(m,m-2,m-4\dots,2-m,2)\,.$$

Hence

$$\delta_G - \delta_K = \frac{1}{2}(m+2)\sum_{k=1}^{m}\widehat{x}_k = \frac{1}{2}(m+2)(1,1,\dots,1,0)\,,$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{4}\,.$$

We now determine the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, < \theta,\delta_K > \ < 0\}\,.$$

Recall that from the proof of lemma 3.2, if Λ is non empty, then any $\theta \in \Lambda$ belongs to $\Phi_G^+ \backslash \Phi_K^+$. It is then easy to verify that the elements of Λ are

$$\widehat{x}_j - \widehat{x}_{m+1}\,,\ \frac{m}{2}+1 \leq j \leq m\,, \qquad < \widehat{x}_j - \widehat{x}_{m+1}, \delta_K >= \frac{1}{2(m+2)}\left(\frac{m}{2} - j\right)\,,$$

$$\widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k\,,\ \frac{m}{2}+2 \leq j \leq m\,, \qquad < \widehat{x}_j + \sum_{k=1}^{m+1} \widehat{x}_k, \delta_K >= \frac{1}{2(m+2)}\left(\frac{m}{2} + 1 - j\right)\,.$$

So

$$\sum_{\theta\in\Lambda} < \theta, \delta_K >= -\frac{m^2}{8(m+2)}\,.$$

Hence, by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{m}{2} - \frac{m^2}{2(m+2)} + \frac{m}{2} = \frac{m+4}{m+2}\,\frac{m}{2}\,.$$

4.3. **Grassmannians $\widetilde{\mathbf{Gr}}_{\mathbf{4}}(\mathbb{R}^{\boldsymbol{m+4}})$, $\boldsymbol{m}$ even $\geq$ 4.** Here $G = \mathrm{Spin}_{m+4}$ and, identifying $\mathbb{R}^m$ with the subspace of $\mathbb{R}^{m+4}$ spanned by $e_1, \ldots e_m$, and $\mathbb{R}^4$ with the subspace spanned by $e_{m+1}, \ldots, e_{m+4}$, K is the subgroup of G defined by

$$\mathrm{Spin}_m\mathrm{Spin}_4 := \left\{\psi \in \mathrm{Spin}_{m+4}\,;\ \psi = \varphi\phi\,,\ \varphi \in \mathrm{Spin}_m\,,\ \phi \in \mathrm{Spin}_4\right\}\,.$$

We consider the common torus of G and K defined by

$$T = \left\{\sum_{k=1}^{\frac{m}{2}+2}\left(\cos(\beta_k) + \sin(\beta_k)\, e_{2k-1}\cdot e_{2k}\right);\ \beta_1, \ldots, \beta_{\frac{m}{2}+2} \in \mathbb{R}\right\}\,.$$

The Lie algebra of T is

$$\mathfrak{T} = \left\{\sum_{k=1}^{\frac{m}{2}+2} \beta_k\, e_{2k-1}\cdot e_{2k}\,;\ \beta_1, \ldots, \beta_{\frac{m}{2}+2} \in \mathbb{R}\right\}\,.$$

We denote by $(x_1, \ldots, x_{\frac{m}{2}+2})$ the basis of $\mathfrak{T}^*$ given by

$$x_k \cdot \sum_{j=1}^{\frac{m}{2}+2} \beta_j\, e_{2j-1}\cdot e_{2j} = \beta_k\,.$$

We introduce the basis $(\widehat{x}_1, \ldots, \widehat{x}_{\frac{m}{2}+2})$ of $i\,\mathfrak{T}^*$ defined by

$$\widehat{x}_k := 2i\, x_k\,, \quad k = 1, \ldots, \frac{m}{2}+2\,.$$

A vector $\mu \in i\,\mathfrak{T}^*$ such that $\mu = \sum_{k=1}^{\frac{m}{2}+2} \mu_k\, \widehat{x}_k$, is denoted by

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{\frac{m}{2}+2})\,.$$

The restriction to $\mathfrak{T}$ of the Killing form B of G is given by

$$B(e_{2k-1}\cdot e_{2k}, e_{2l-1}\cdot e_{2l}) = -8\,(m+2)\,\delta_{kl}\,.$$

It is easy to verify that the scalar product on $i\,\mathfrak{T}^*$ induced by the Killing form sign changed is given by

$$\forall \mu = (\mu_1, \ldots, \mu_{\frac{m}{2}+2}) \in i\,\mathfrak{T}^*\,,\, \forall \mu' = (\mu'_1, \ldots, \mu'_{\frac{m}{2}+2}) \in i\,\mathfrak{T}^*\,,$$

(15)
$$< \mu, \mu' > = \frac{1}{2(m+2)} \sum_{k=1}^{\frac{m}{2}+2} \mu_k\, \mu'_k\,.$$

Considering the decomposition of the complexified Lie algebra of G under the action of T, it is easy to verify that T is a common maximal torus of G and K, and that the respective roots are given by

$$\pm\,(\widehat{x}_i + \widehat{x}_j)\,,\ \pm(\widehat{x}_i - \widehat{x}_j)\,,\ 1 \leq i < j \leq \frac{m}{2} + 2\,, \qquad \text{for } G\,,$$

$$\begin{cases} \pm(\widehat{x}_i + \widehat{x}_j)\,,\, \pm(\widehat{x}_i - \widehat{x}_j)\,,\, 1 \leq i < j \leq \frac{m}{2} \\ \pm(\widehat{x}_{\frac{m}{2}+1} + \widehat{x}_{\frac{m}{2}+2})\,,\, \pm(\widehat{x}_{\frac{m}{2}+1} - \widehat{x}_{\frac{m}{2}+2})\,, \end{cases} \qquad \text{for } K\,.$$

We consider as sets of positive roots

$$\Phi_G^+ = \left\{ \widehat{x}_i + \widehat{x}_j\,,\, \widehat{x}_i - \widehat{x}_j\,,\ 1 \leq i < j \leq \frac{m}{2} + 2 \right\}\,,$$

and

$$\Phi_K^+ = \left\{ \widehat{x}_i + \widehat{x}_j\,,\, \widehat{x}_i - \widehat{x}_j\,,\ 1 \leq i < j \leq \frac{m}{2}\,,\ \widehat{x}_{\frac{m}{2}+1} + \widehat{x}_{\frac{m}{2}+2}\,,\, \widehat{x}_{\frac{m}{2}+1} - \widehat{x}_{\frac{m}{2}+2} \right\}\,.$$

Then

$$\delta_G = \sum_{k=1}^{\frac{m}{2}+2} (\frac{m}{2} + 2 - k)\, \widehat{x}_k = (\frac{m}{2} + 1, \frac{m}{2}, \ldots, 1, 0)\,,$$

and

$$\delta_K = \sum_{k=1}^{\frac{m}{2}} (\frac{m}{2} - k)\, \widehat{x}_k + \widehat{x}_{\frac{m}{2}+1} = (\frac{m}{2} - 1, \frac{m}{2} - 2, \ldots, 1, 0)\,.$$

Hence

$$\delta_G - \delta_K = 2 \sum_{k=1}^{\frac{m}{2}} \widehat{x}_k = 2\,(1, 1, \ldots, 1, 0, 0)\,,$$

so

$$\|\delta_G - \delta_K\|^2 = \frac{m}{m+2}\,.$$

On the other hand, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, < \theta, \delta_K > \ < 0\}\,,$$

has only one element, namely

$$\widehat{x}_{\frac{m}{2}} - \widehat{x}_{\frac{m}{2}+1}\,,\ \text{with} \quad < \widehat{x}_{\frac{m}{2}} - \widehat{x}_{\frac{m}{2}+1}, \delta_K >= -1\,.$$

Hence, by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{2m}{m+2} - \frac{2}{m+2} + \frac{m}{2} = \frac{m^2 + 6m - 4}{2(m+2)}\,.$$

4.4. **The four exceptional cases.** Note first that since all the groups G we consider are simple, their roots system are irreducible so, up to a constant, there is only one W_G-invariant scalar product on the subspace generated by the set of roots, cf. for instance Remark (5.10), § V in [BtD85].
We use the description of root systems given in [BMP85]. Those root systems are expressed in the simple root basis (α_i). Note that the W_G-invariant scalar product $(\,,\,)$ used there is such that $(\alpha, \alpha) = 2$ for any long root α. In order to compare it with the scalar product $<,>$ induced by the Killing form sign-changed, we use the "strange formula" of Freudenthal and de Vries, (cf. 47-11 in [FdV69]):

$$< \delta_G, \delta_G >= \frac{1}{24} \dim G\,. \tag{16}$$

To determine the set of K-positive roots, we use theorem 13, theorem 14 and the proof of theorem 18 in [CG88]. By those results, the set Φ_K^+ may be defined as follows. Let $\theta = \sum m_i\, \alpha_i$ be the highest root. In all cases considered, there exists an index j such that $m_j = 2$. Then

$$\Phi_K^+ = \left\{ \sum n_i \alpha_i\,;\, n_j \neq 1 \right\}\,.$$

4.4.1. *The symmetric space* $\mathrm{G}_2/\mathrm{SO}_4$. Using the results of pages 18 and 64 in [BMP85], we get

$$\delta_G = 3\,\alpha_1 + 5\,\alpha_2\,.$$

By the expression of the Cartan matrix, the scalar product matrix is, in the basis (α_1, α_2), $\begin{pmatrix} 2 & -1 \\ -1 & 2/3 \end{pmatrix}$, hence

$$\|\delta_G\|^2_{(\,,\,)} = \frac{14}{3}\,.$$

On the other hand, by the formula of Freudenthal and de Vries,

$$\|\delta_G\|^2_{<\,,\,>} = \frac{7}{12}\,,$$

so

$$<\,,\,>= \frac{1}{8}\,(\,,\,)\,.$$

The set of K-positive roots is

$$\Phi_K^+ = \{2\,\alpha_1 + 3\,\alpha_2, \alpha_2\}\,,$$

hence

$$\delta_K = \alpha_1 + 2\,\alpha_2\,,$$

so

$$\delta_G - \delta_K = 2\,\alpha_1 + 3\,\alpha_2\,.$$

Hence

$$\|\delta_G - \delta_K\|^2_{<\,,\,>} = \frac{1}{8}\,\|\delta_G - \delta_K\|^2_{(\,,\,)} = \frac{1}{4}\,.$$

Finally, it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, < \theta, \delta_K > \, < 0\}\,,$$

is empty, hence by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{1}{2} + 1 = \frac{3}{2}\,.$$

4.4.2. *The symmetric space* $\mathrm{E}_6/(\mathrm{SU}_6\mathrm{SU}_2)$. Using the results of pages 14 and 60 in [BMP85], we get

$$\delta_G = 8\,\alpha_1 + 15\,\alpha_2 + 21\,\alpha_3 + 15\,\alpha_4 + 8\,\alpha_5 + 11\,\alpha_6\,.$$

Since all roots have same length equal to 2, we may introduce the fundamental weight basis (ω_i) because

$$(\omega_i, \alpha_j) = \delta_{ij}\,.$$

Since $\delta_G = \sum \omega_i$, we get

$$\|\delta_G\|^2_{(\,,\,)} = 78\,,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|^2_{<\,,\,>} = \frac{78}{24}\,,$$

so

$$<\,,\,> = \frac{1}{24}\,(\,,\,)\,.$$

The set of K-positive roots may be defined by

$$\Phi_K^+ = \left\{\sum_{i=1}^{6} n_i\,\alpha_i\,;\, n_6 \neq 1\right\}\,.$$

Then

$$\begin{aligned}\delta_K &= 3\,\alpha_1 + 5\,\alpha_2 + 6\,\alpha_3 + 5\,\alpha_4 + 3\,\alpha_5 + \alpha_6\\ &= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 - 4\,\omega_6\,.\end{aligned}$$

Hence

$$\delta_G - \delta_K = 5\,\alpha_1 + 10\,\alpha_2 + 15\,\alpha_3 + 10\,\alpha_4 + 5\,\alpha_5 + 10\,\alpha_6 = 5\,\omega_6\,.$$

So

$$\|\delta_G - \delta_K\|^2_{<\,,\,>} = \frac{1}{24}\,\|\delta_G - \delta_K\|^2_{(\,,\,)} = \frac{25}{12}\,.$$

On the other hand it is easy to verify that the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, <\theta, \delta_K> \,< 0\}\,,$$

has 7 elements and that

$$\sum_{\theta\in\Lambda} <\theta, \delta_K> = \frac{1}{24}\sum_{\theta\in\Lambda}(\theta, \delta_K) = -\frac{7}{12}\,.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{50}{12} - \frac{28}{12} + 5 = \frac{41}{6}\,.$$

4.4.3. *The symmetric space* $\mathrm{E}_7/(\mathrm{Spin}_{12}\mathrm{SU}_2)$. By the results of pages 15 and 61 in [BMP85], we get

$$\delta_G = \frac{1}{2}\,(34\,\alpha_1 + 66\,\alpha_2 + 96\,\alpha_3 + 75\,\alpha_4 + 52\,\alpha_5 + 27\,\alpha_6 + 49\,\alpha_7)\,.$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis (ω_i). We get

$$\|\delta_G\|^2_{(\,,\,)} = \frac{399}{2}\,,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|^2_{<\,,\,>} = \frac{133}{24}\,,$$

so

$$<\,,\,>= \frac{1}{36}\,(\,,\,)\,.$$

The set of K-positive roots may be defined by

$$\Phi_K^+ = \left\{\sum_{i=1}^{7} n_i\,\alpha_i\,;\, n_1 \neq 1\right\}\,.$$

Then

$$\begin{aligned}\delta_K &= \frac{1}{2}\,(2\,\alpha_1 + 18\,\alpha_2 + 32\,\alpha_3 + 27\,\alpha_4 + 20\,\alpha_5 + 11\,\alpha_6 + 17\,\alpha_7)\\ &= -7\,\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7\,.\end{aligned}$$

Hence

$$\delta_G - \delta_K = 16\,\alpha_1 + 24\,\alpha_2 + 32\,\alpha_3 + 24\,\alpha_4 + 16\,\alpha_5 + 8\,\alpha_6 + 16\,\alpha_7 = 8\,\omega_6\,.$$

So

$$\|\delta_G - \delta_K\|^2_{<\,,\,>} = \frac{1}{36}\,\|\delta_G - \delta_K\|^2_{(\,,\,)} = \frac{32}{9}\,.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, <\theta, \delta_K>\; < 0\}\,,$$

has 13 elements and that

$$\sum_{\theta\in\Lambda} <\theta, \delta_K>= \frac{1}{36}\sum_{\theta\in\Lambda}(\theta, \delta_K) = -\frac{41}{36}\,.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{64}{9} - \frac{41}{9} + 8 = \frac{95}{9}\,.$$

4.4.4. *The symmetric space* $E_8/(E_7SU_2)$. By the results of pages 16, 62 and 63 in [BMP85], we get

$$\delta_G = 29\,\alpha_1 + 57\,\alpha_2 + 84\,\alpha_3 + 110\,\alpha_4 + 135\,\alpha_5 + 91\,\alpha_6 + 46\,\alpha_7 + 68\,\alpha_8\,.$$

Here again, since all roots have same length equal to 2, we may consider the fundamental weight basis (ω_i). We get

$$\|\delta_G\|^2_{(\,,\,)} = 620\,,$$

whereas by the formula of Freudenthal and de Vries,

$$\|\delta_G\|^2_{<\,,\,>} = \frac{248}{24} = \frac{31}{3}\,,$$

so

$$<\,,\,> = \frac{1}{60}\,(\,,\,)\,.$$

The set of K-positive roots may be defined by

$$\Phi_K^+ = \left\{ \sum_{i=1}^{8} n_i\,\alpha_i\,;\, n_1 \neq 1 \right\}.$$

Then

$$\begin{aligned}\delta_K &= \alpha_1 + 15\,\alpha_2 + 28\,\alpha_3 + 40\,\alpha_4 + 51\,\alpha_5 + 35\,\alpha_6 + 18\,\alpha_7 + 26\,\alpha_8\\ &= -13\,\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8\,.\end{aligned}$$

Hence

$$\delta_G - \delta_K = 28\,\alpha_1 + 42\,\alpha_2 + 56\,\alpha_3 + 70\,\alpha_4 + 84\,\alpha_5 + 56\,\alpha_6 + 28\,\alpha_7 + 42\,\alpha_8 = 14\,\omega_6\,.$$

So

$$\|\delta_G - \delta_K\|^2_{<\,,\,>} = \frac{1}{60}\,\|\delta_G - \delta_K\|^2_{(\,,\,)} = \frac{98}{15}\,.$$

On the other hand it can be verified that the set

$$\Lambda := \{\theta \in \Phi_G^+\,;\, <\theta, \delta_K> \; < 0\}\,,$$

has 25 elements and that

$$\sum_{\theta\in\Lambda} <\theta, \delta_K> = \frac{1}{60} \sum_{\theta\in\Lambda} (\theta, \delta_K) = -\frac{137}{60}\,.$$

So by formula (5), the square of the first eigenvalue λ of the Dirac operator is given by

$$\lambda^2 = \frac{196}{15} - \frac{137}{15} + 14 = \frac{269}{15}\,.$$

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Laboratoire Jean Leray, UMR CNRS 6629, Département de Mathématiques, Université de Nantes, 2, rue de la Houssinière, BP 92208, F-44322 NANTES CEDEX 03

E-mail address: milhorat@math.univ-nantes.fr